



NE 112 *Linear algebra for nanotechnology engineering*

1.6.2 Introducing j

Douglas Wilhelm Harder, LEL, M.Math.

dwharder@uwaterloo.ca

dwharder@gmail.com





Outline

- In this topic, we will
 - Introduce the justification for the imaginary unit j





Square roots

- By definition, $\sqrt{x}$ is any number such that

$$\left(\sqrt{x}\right)^2 = x$$

- Recall that $\sqrt{2x} = \sqrt{2}\sqrt{x}$ as $2 \geq 0$ and $\left(\sqrt{2}\sqrt{x}\right)^2 = \sqrt{2}^2 \sqrt{x}^2 = 2x$
 - Consequently, if $c \geq 0$, we can always write

$$\sqrt{cx} = \sqrt{c}\sqrt{x}$$

- Note this is not true for anything that is not zero or a positive real:

$$1 = \sqrt{1} = \sqrt{-1 \cdot -1} \neq \sqrt{-1}\sqrt{-1} = \left(\sqrt{-1}\right)^2 = -1$$





Dealing with $\sqrt{-1}$

- Consequently, we note that theoretically, we could have:

$$\sqrt{-4} = \sqrt{4}\sqrt{-1} = 2\sqrt{-1}$$

- Suppose we just assume that $\sqrt{-1}$ is a value, so

$$\left(\sqrt{-1}\right)^2 = -1$$

- We also note that

$$\left(-\sqrt{-1}\right)^2 = (-1)^2 \left(\sqrt{-1}\right)^2 = 1 \cdot \left(\sqrt{-1}\right)^2 = -1$$





Dealing with $\sqrt{-1}$

- Thus we can have the product $\beta\sqrt{-1}$
 - Could add to this a real number α ?

$$\alpha + \beta\sqrt{-1}$$

- After all, given x , we could add $\alpha + \beta x$
- We could thus calculate the following using *foil*:
$$\begin{aligned}(\alpha + \beta\sqrt{-1})(\alpha + \beta\sqrt{-1}) &= \alpha^2 + \alpha\beta\sqrt{-1} + \alpha\beta\sqrt{-1} + (\beta\sqrt{-1})^2 \\ &= \alpha^2 - \beta^2 + 2\alpha\beta\sqrt{-1}\end{aligned}$$





Are these roots?

- Now, consider the polynomial $z^2 + 1$
 - If we substitute into this polynomial

$$z = \sqrt{-1}$$

$$\left(\sqrt{-1}\right)^2 + 1 = -1 + 1 = 0$$





Are these roots?

- Now, consider the polynomial $z^2 + 2z + 5$
 - If we substitute into this polynomial

$$z = -1 + 2\sqrt{-1}$$

$$\begin{aligned} & \left(-1 + 2\sqrt{-1}\right)^2 + 2\left(-1 + 2\sqrt{-1}\right) + 5 \\ &= (-1)^2 + 2(-1)(2\sqrt{-1}) + (2\sqrt{-1})^2 - 2 + 4\sqrt{-1} + 5 \\ &= 1 - 4\sqrt{-1} + 4(-1) - 2 + 4\sqrt{-1} + 5 \\ &= 1 - 4 - 2 + 5 + (-4\sqrt{-1} + 4\sqrt{-1}) \\ &= 0 \end{aligned}$$





Quadratic formula?

- Does the quadratic formula give the same for $z^2 + 1$?

$$\begin{aligned}\frac{-0 \pm \sqrt{0 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} &= \pm \frac{\sqrt{-4}}{2} \\ &= \pm \frac{\sqrt{4} \sqrt{-1}}{2} \\ &= \pm \frac{2\sqrt{-1}}{2} = \pm \sqrt{-1}\end{aligned}$$





Quadratic formula?

- Does the quadratic formula give the same for $z^2 + 2z + 5$?

$$\begin{aligned}\frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 5}}{2 \cdot 1} &= \frac{-2}{2} \pm \frac{\sqrt{-16}}{2} \\ &= -1 \pm \frac{\sqrt{16}\sqrt{-1}}{2} \\ &= -1 \pm \frac{4\sqrt{-1}}{2} = -1 \pm 2\sqrt{-1}\end{aligned}$$





Introducing j

- Issue: It is annoying to always write down $\sqrt{-1}$
 - Thus, mathematicians introduce

$$i \stackrel{\text{def}}{=} \sqrt{-1}$$

- Problem: Engineers use i for current...

$$j \stackrel{\text{def}}{=} \sqrt{-1}$$

- Thus, we write $z = -1 + 2j$ instead of $z = -1 + 2\sqrt{-1}$ or $z = -1 + \sqrt{-4}$
 - If a mathematician ever complains about j , just say:
“The field $a + ib$ is a subfield of the quaternions,
and the field $a + jb$ is another subfield that is isometric to yours.”





Introducing j

- Thus, the roots of the polynomial $z^2 + 1$ appear to be

$$z = \pm j$$

- Similarly, the roots of the polynomial $z^2 + 2z + 5$ appear to be

$$z = -1 \pm 2j$$





Introducing j

- Now, given the definition, $j^2 = -1$:
 - What is j^3 ?

$$\begin{aligned}j^3 &= j^2 \cdot j \\&= (-1) \cdot j \\&= -j\end{aligned}$$

- What is j^4 ?

$$\begin{aligned}j^4 &= j^2 \cdot j^2 \\&= (-1) \cdot (-1) \\&= 1\end{aligned}$$

- What is $j^{1000001}$?

$$\begin{aligned}j^{1000001} &= j^{4 \cdot 250000 + 1} \\&= j^{4 \cdot 250000} \cdot j^1 \\&= (j^4)^{250000} \cdot j \\&= 1^{250000} \cdot j = j\end{aligned}$$



Introducing j

- Also note that,

$$\begin{aligned}j(-j) &= -j^2 \\ &= -(-1) = 1\end{aligned}$$

- Thus, $j^{-1} = -j$



Introducing j

- Let's have some more fun:

$$\begin{aligned}(1+j)(1+j) &= 1 + j + j + j^2 \\ &= 1 + 2j - 1 = 2j\end{aligned}$$

- Does this not mean a square root of $2j$ is $1+j$?
 - If z is a square root of another number, is not also $-z$?

$$\begin{aligned}(-1-j)(-1-j) &= 1 + j + j + j^2 \\ &= 1 + 2j - 1 = 2j\end{aligned}$$



Summary

- In this topic, you now
 - Understand where j comes from and what it defines
 - Know that some polynomials that do not have real roots appear to have roots if we allow the use of j





References

- [1] https://en.wikipedia.org/wiki/Imaginary_unit



Acknowledgments

None so far.



Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas. Mathematical equations are prepared in MathType by Design Science, Inc. Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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